Some Aspects of the Boltzmann Equation for a Granular Gas

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Abstract. The Boltzmann equation for a gas of smooth, inelastic hard spheres is introduced and its homogeneous solution for an isolated system is discussed. The possibility of hydrodynamic excitations is explored using the linearized Boltzmann equation for small spatial perturbations of the homogeneous state. It is shown that the spectrum of the generator for this linear dynamics contains five points at long wavelengths that are the origin of hydrodynamic excitations. With this knowledge, response functions are introduced that allow the formal derivation of linear hydrodynamic equations and associated Green-Kubo expressions for the transport coefficients at Navier-Stokes order.

1. INTRODUCTION

One of the most important characteristics of a real granular gas is hard binary *inelastic* collisions. This feature of inelasticity alone is responsible for many of the qualitative differences between granular and normal gases. A simple model to study these differences is therefore a system of N smooth, hard spheres undergoing inelastic collisions. With the collision rule specified (see below) the trajectories in the N particle phase space are well-defined and the formal structure of non-equilibrium statistical mechanics can be applied [1, 2, 3]. In particular, the BBGKY hierarchy for reduced distribution functions can be obtained. At sufficiently low density small parameter expansions developed for normal gases can be applied for an approximate closure of the hierarchy equations [4, 5]. To leading order in the ratio of the diameter of the spheres to the mean free path, the Boltzmann equation for the single particle reduced distribution function is obtained[1]. The objective here is to explore some of the consequences of inelasticity for solutions to the Boltzmann equation. Specifically, the existence of hydrodynamic excitations for small spatial perturbations at long wavelengths is demonstrated [6, 7] and explored to Navier-Stokes order.

The Boltzmann equation is introduced in the next section, parameterized by a single scalar constant - the restitution coefficient α - differentiating granular gases ($\alpha < 1$) from normal gases ($\alpha = 1$). For an isolated system (no external driving sources) it is noted that there is no stationary solution due to the continual loss of energy in binary collisions. However, there is a "universal" solution analogous to the Maxwellian whose time dependence occurs entirely through a second moment of the distribution (the "temperature"). This solution is special in the sense that it appears that general homogenous states relax after a few mean collisions towards this solution. This is similar to the rapid velocity relaxation to a Maxwell distribution in normal gases. A great deal is known about this special solution, the "homogeneous cooling solution" (HCS), from both analytic [8, 9, 10] and Monte Carlo simulation studies [11].

In section 3 small deviations from the HCS due to spatial perturbations are described in terms of the linearized Boltzmann equation. It is shown that the generator for this linear dynamics has a spectrum containing five points corresponding to hydrodynamic excitations at long wavelengths. The eigenfunctions are identified in terms of derivatives of the HCS. These eigenvalues and eigenfunctions show the existence of hydrodynamic modes for granular gases. Next, hydrodynamic response functions are defined from the formal solution to the linearized Boltzmann equation. The hydrodynamic spectrum for these response functions is obtained to Navier-Stokes order [6, 7], and the transport coefficients identified in the form of Green-Kubo expressions [12, 13, 6].

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[1] Important questions remain about the size of corrections to the Boltzmann limit due to velocity correlations, at any given value of this small parameter.

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2. BOLTZMANN EQUATION AND HOMOGENEOUS COOLING SOLUTION

Consider a system of N smooth hard particles in a large volume V. The particles move in a straight line until a pair is at contact. The velocities of that pair are then changed instantaneously according to a prescribed inelastic scattering law. In general this law can be quite complex, including both normal and tangential friction, the shape of the particles, and a velocity dependent normal restitution coefficient representing the distortion of real grains on collision. After the collision the particles again move on straight lines with the new velocities until another pair is at contact. In this way a deterministic trajectory in the N particle phase space is defined. A probability density in this phase space can be introduced to represent incomplete knowledge of the initial state, and all the elements of a non-equilibrium statistical mechanics are then available for analysis of this system [1]. In particular, a Liouville equation for the probability density is obtained and from it follows a BBGKY hierarchy for the s-particle reduced distribution functions. The first of these equations is the precursor for the Boltzmann equation for appropriate conditions of low density.

To simplify the discussion all subsequent analysis will be for smooth, inelastic hard spheres of diameter σ. The Liouville dynamics for the distribution functions is the inverse of that for the trajectories of the particles and involves "restituting" collisions for the pair i, j given by

$$\mathbf{v}_i' \equiv \mathbf{v}_i - \frac{1+\alpha}{2\alpha}(\mathbf{g}_{ij} \cdot \hat{\sigma})\hat{\sigma}, \qquad \mathbf{v}_j' \equiv \mathbf{v}_j + \frac{1+\alpha}{2\alpha}(\mathbf{g}_{ij} \cdot \hat{\sigma})\hat{\sigma}, \tag{1}$$

where $\hat{\sigma}$ is a unit vector along the direction of the relative position $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$, and $\mathbf{g}_{ij} \equiv \mathbf{v}_i - \mathbf{v}_j$ is the relative velocity. The restitution coefficient α takes the values $0 < \alpha \le 1$, where $\alpha = 1$ corresponds to elastic collisions. As noted above, it is the single parameter differentiating granular gases from normal gases in this representation. The probability density, $f(\mathbf{r}_1, \mathbf{v}_1, t)$, for the position and velocity in the single particle phase space obeys the exact first BBGKY hierarchy equation

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_1 \cdot \nabla_1\right) f(\mathbf{r}_1, \mathbf{v}_1, t) = \sigma^2 \int d\mathbf{v}_2 \int d\hat{\sigma}\,\Theta(\hat{\sigma} \cdot \mathbf{g})(\hat{\sigma} \cdot \mathbf{g}) \left\{ \alpha^{-2} f^{(2)}(\mathbf{r}_1, \mathbf{r}_1 - \sigma, \mathbf{v}_1', \mathbf{v}_2', t) \right.$$
$$\left. - f^{(2)}(\mathbf{r}_1, \mathbf{r}_1 + \sigma, \mathbf{v}_1, \mathbf{v}_2, t) \right\}. \tag{2}$$

Here $f^{(2)}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{v}_1, \mathbf{v}_2, t)$ is the joint probability density in the two particle phase space. It has been shown elsewhere [4] that a dimensionless form of the BBGKY hierarchy can be expressed in terms of the single parameter $\lambda \equiv \sigma/\ell$, the ratio of the "force range" to the mean free path $\ell \equiv 1/(n\sigma^2)$. This parameter is small at low density, suggesting an expansion for a solution to the entire hierarchy as a power series in λ. In particular, this gives

$$f^{(2)}(\mathbf{r}_1, \mathbf{r}_1 \pm \sigma, \mathbf{v}_1, \mathbf{v}_2, t) \to f(\mathbf{r}_1, \mathbf{v}_1, t) f(\mathbf{r}_1, \mathbf{v}_2, t) + O\lambda \tag{3}$$

and the first BBGKY hierarchy equation to this order becomes the Boltzmann equation

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right) f(\mathbf{r}, \mathbf{v}, t) = J(\mathbf{v} \mid f, f) \tag{4}$$

where the nonlinear Boltzmann collision operator is identified as

$$J(\mathbf{v} \mid f, f) = \sigma^2 \int d\mathbf{v}_2 \int d\hat{\sigma}\,\Theta(\hat{\sigma} \cdot \mathbf{g})(\hat{\sigma} \cdot \mathbf{g}) \left(\alpha^{-2} f(\mathbf{r}, \mathbf{v}_1', t) f(\mathbf{r}, \mathbf{v}_2', t) - f(\mathbf{r}_1, \mathbf{v}_1, t) f(\mathbf{r}_2, \mathbf{v}_2, t) \right) \tag{5}$$

This formal expansion of the hierarchy also gives an equation for the corrections to the Boltzmann equation. Therefore conceptual issues regarding the validity and context of the Boltzmann equation for granular gases can be studied quantitatively [14, 5]. This topic will not be explored here.

An important difference between normal and granular gases is the absence of energy conservation in the two particle collision. This is reflected in the moments of the collision operator which can be calculated directly from the above definitions

$$\int d\mathbf{v} \begin{pmatrix} 1 \\ \mathbf{v} \\ \frac{1}{2}m(\mathbf{v} - \mathbf{u})^2 \end{pmatrix} J(\mathbf{v}|f, f) = \begin{pmatrix} 0 \\ \mathbf{0} \\ -\frac{3}{2}nT\xi \end{pmatrix}, \tag{6}$$

The two zeros on the right side of (4) correspond to conservation of mass and momentum. The last term results from non-conservation of energy and ξ is the fractional rate of decrease in the energy due to the inelasticity. The kinetic "temperature" T and flow velocity $\mathbf{u}$ are defined in the usual way

$$\left(\begin{array}{c} \frac{3}{2}nT \\ n\mathbf{u} \end{array} \right) = \int d\mathbf{v} \left(\begin{array}{c} \frac{1}{2}m\left(\mathbf{v}-\mathbf{u}\right)^2 \\ \mathbf{v} \end{array} \right) f(\mathbf{r},\mathbf{v},t,). \tag{7}$$

The "cooling rate" $\xi[f,f]$ is a positive functional of f. This implies that the energy of an isolated system decreases monotonically. Consequently, there is no stationary state such as the Maxwell-Boltzmann distribution for normal gases. Instead, there is a homogeneous cooling solution with the scaling form

$$f_{hcs}\left(\mathbf{v},t\left(t\right)\right) = nv_0^{-3}(T(t))\phi\left(v/v_0(T(t))\right), \qquad v_0(T(t)) = \sqrt{2T(t)/m}. \tag{8}$$

An equation for the temperature follows directly from this scaling and the Boltzmann equation without further assumption

$$\left(\partial_t + \xi\left(T\left(t\right)\right)\right)T\left(t\right) = 0. \tag{9}$$

where

$$\xi\left(T\left(t\right)\right) = \xi[f_{hcs},f_{hcs}] = \left(1-\alpha^2\right)\frac{m\pi\sigma^2}{24nT}\int d\mathbf{v}\int d\mathbf{v}_1 \left|\mathbf{v}-\mathbf{v}_1\right|^3 f_{hcs}\left(v\right)f_{hcs}\left(v_1\right)). \tag{10}$$

The temperature dependence of ξ can be made explicit through the scaling form of f_{hcs}, so (9) can be integrated for the exact time dependence of the temperature. The function f_{hcs} is determined from the Boltzmann equation which now takes the form

$$\frac{1}{2}\xi\nabla_{\mathbf{v}}\cdot\left(\mathbf{v}f_{hcs}\right) = J\left(\mathbf{v}|f_{hcs},f_{hcs}\right). \tag{11}$$

This equation must be solved self-consistently with the equation for ξ to determine the HCS. Although an explicit solution has not yet been obtained, its form at large and small velocities has been determined by good analytic approximations [8, 9]. More generally the universality of the HCS as the attractor for a wide class of initial conditions, its functional form, and the scaling property have been verified by Monte Carlo simulation [11]. In all of the following the HCS will be treated as a known function.

3. LINEAR BOLTZMANN EQUATION AND HYDRODYNAMIC MODES

As noted above, there is evidence that small homogeneous perturbations of the HCS decay on the order of the time for a few collisions. A description of the response to small *spatial* perturbations of the HCS is both more interesting and more complex, and is the object of the remaining discussion here. The simplest case is an *initial* spatial perturbation of the HCS (other examples might arise from small external forces or boundery condtions). The distribution function is written as

$$f\left(\mathbf{r},\mathbf{v},t\right) = f_{hcs}\left(\mathbf{v},v\left(t\right)\right)\left[1 + e^{i\mathbf{k}\cdot\mathbf{r}}\Delta\left(\mathbf{k},\mathbf{v},t\right)\right] \tag{12}$$

and solutions are sought to linear order in Δ. Since only linear terms are retained, it is sufficient to consider only a single Fourier mode. For the moment, questions of stability are deferred. The Boltzmann equation then takes a linear form for Δ. Due to the time dependence of the reference state this equation depends explicitly on time. It is convenient, therefore, to introduce the dimensionless variables

$$f\left(\mathbf{v},t\right) = nv_0^{-3}(T(t))f^*\left(\mathbf{v}^*,s\right), \qquad \Delta^*\left(\mathbf{k}^*,\mathbf{v}^*,s\right) = \Delta\left(\mathbf{k},\mathbf{v},t\right), \qquad \mathbf{k}^* = \mathbf{k}\ell$$

$$s = \int_0^t dt'\frac{v_0(T(t))}{\ell}, \qquad \mathbf{v}^* = \frac{\mathbf{v}}{v_0(T(t))}, \qquad \xi^* = \frac{\ell\xi}{v_0(T(t))} \tag{13}$$

where $\ell = 1/n\sigma^3$ is the mean free path and s is the average number of collisions in the time interval $(0,t)$. The linearized Boltzmann equation then becomes

$$\left(\partial_s + i\mathbf{k}^*\cdot\mathbf{v}^* + \Lambda\right)\Delta^* = 0 \tag{14}$$

where the linear collision operator Λ is defined by

$$\Lambda X = LX + \frac{1}{2}\xi^* f_{hcs}^{-1} \nabla_{\mathbf{v}} \cdot \left(\mathbf{v} f_{hcs} X\right) \tag{15}$$

In these dimensionless variables Λ is time independent. The first term on the right is the linearized Boltzman collision operator

$$LX = -\frac{\ell}{v_0(T(t)) f_{hcs}} \left[J(\mathbf{v} \mid f_{hcs}, f_{hcs}X) + J(\mathbf{v} \mid f_{hcs}X, f_{hcs}) \right]$$

while the second term describes the effects of cooling explicitly, and arises because the derivative with respect to s is taken at fixed $\mathbf{v}^*$.

The formal solution to the linear Boltzmann equation is

$$\Delta^* \left(\mathbf{k}^*, \mathbf{v}^*, s\right) = e^{(i\mathbf{k}^* \cdot \mathbf{v}^* + \Lambda)s} \Delta^* \left(\mathbf{k}^*, \mathbf{v}^*, 0\right). \tag{16}$$

This shows that all linear excitations of the granular gas are determined from the spectrum of $i\mathbf{k}^* \cdot \mathbf{v}^* + \Lambda$ (the relevant Hilbert space is defined in the next section)

$$\left(i\mathbf{k}^* \cdot \mathbf{v}^* + \Lambda\right) \psi_n(\mathbf{k}^*) = \lambda_n(k) \psi_n(\mathbf{k}^*). \tag{17}$$

In particular, if a hydrodynamic description is applicable to granular gases the corresponding excitations must occur in this spectrum. For a normal gas with elastic collisions the hydrodynamic spectrum is defined in terms of the $\mathbf{k}^* = 0$ spectrum. In that case there are five eigenfunctions (the summational invariants) with zero eigenvalue, corresponding to the conservation of mass, energy, and momentum. The hydrodynamic spectrum for $\mathbf{k}^* \neq 0$ is that which is continuously connected to the five $\mathbf{k}^* = 0$ zero eigenvalues.

A similar analysis applies for granular gases. The hydrodynamic eigenvalues for $\mathbf{k}^* = 0$ can be identified by differentiating the equation for the HCS with respect to the average density, temperature, and flow velocity (denoted generically by y_γ) [6, 7]

$$\frac{\partial}{\partial y_\gamma} \left[\frac{1}{2} \xi \nabla_{\mathbf{v}} \cdot \left(\mathbf{v} f_{hcs}\right) - J\left(\mathbf{v} | f_{hcs}, f_{hcs}\right) \right] = 0. \tag{18}$$

In this way five eigenfunctions and eigenvalues are obtained

$$\Lambda \phi_n = \lambda_n(0) \phi_n, \qquad \phi_n = \psi_n(0) \tag{19}$$

$$\lambda_n(0) \Leftrightarrow \left(0, \frac{1}{2}\xi^*, -\frac{1}{2}\xi^*, -\frac{1}{2}\xi^*, -\frac{1}{2}\xi^*\right) \tag{20}$$

$$\phi_n \Leftrightarrow \left(4 + v^* \partial_{v^*} \ln f_{hcs}^*, -3 - v^* \partial_{v^*} \ln f_{hcs}^*, -\nabla_{\mathbf{v}^*} \ln f_{hcs}^*\right). \tag{21}$$

These are the hydrodynamic modes since the eigenvalues are the same as those of the exact macroscopic balance equations for mass, energy, and momentum, linearized and taken at $k^* = 0$. This is therefore a direct confirmation of hydrodynamic excitations for a granular gas, independent of other derivations such as the Chapman-Enskog expansion. Of course there may be other non-hydrodynamics solutions for $\lambda(0)$, but to simplify the notation it will be understood below that $\lambda(0)$ refers specifically to the set of five values above. For small but finite wavevectors, $\mathbf{k}^* \neq 0$, the corresponding eigenfunctions and eigenvalues can be calculated from (17) by perturbation theory using (19) - (21) as the reference state. The results obtained in this way to Navier-Stokes order agree in detail with those from the Chapman-Enskog expansion [6, 7]. Instead, an alternative approach based on response functions will be explored here.

4. HYDRODYNAMIC RESPONSE FUNCTIONS

To simplify the notation it is useful to introduce a scalar product

$$(a, b) = \int d\mathbf{v}^* f_{hcs}^* (v^*) a^+ (\mathbf{v}^*) b(\mathbf{v}^*) \tag{22}$$

and a set of functions biorthogonal to the $\mathbf{k}^* = 0$ eigenfunctions

$$\widetilde{\phi}_n \Leftrightarrow \left(1, \left(\frac{1}{3}v^{*2} + \frac{1}{2}\right), \mathbf{v}^*\right), \qquad \left(\widetilde{\phi}_n, \phi_m\right) = \delta_{nm} \tag{23}$$

Now consider an initial perturbation of the HCS in terms of the eigenfunctions ϕ_m

$$\Delta^*\left(\mathbf{k}^*, \mathbf{v}^*, 0\right) = \phi_m\left(\mathbf{v}^*\right) \delta y_m(\mathbf{k}, 0). \tag{24}$$

The coefficients $\delta y_m(\mathbf{k}^*, 0)$ can be determined by calculating the average of $\widetilde{\phi}_n$,

$$\left(\widetilde{\phi}_n, \Delta^*\left(0\right)\right) = \delta y_n(\mathbf{k}^*, 0). \tag{25}$$

Since the $\widetilde{\phi}_n$ are linear combinations of the summational invariants, the $\delta y_n(\mathbf{k}, 0)$ are linear combinations of the inital deviations for the density, temperature, and flow velocity from their values in the HCS. Thus the initial state is a hydrodynamic perturbation. Their response at some later time is therefore

$$\delta y_n(\mathbf{k}^*, s) = G_{nm}(\mathbf{k}^*, s) \delta y_m(\mathbf{k}^*, 0). \tag{26}$$

The response functions $G_{nm}(\mathbf{k}^*, s)$ are given by

$$G_{nm}(\mathbf{k}^*, s) = \left(\widetilde{\phi}_n, \phi_m(\mathbf{k}^*, s)\right) = \left(\widetilde{\phi}_n(\mathbf{k}^*, s), \phi_m\right), \tag{27}$$

where the time dependent functions are

$$\phi_m(\mathbf{k}^*, s) = e^{-(i\mathbf{k}^* \cdot \mathbf{v}^* + \Lambda)s} \phi_m, \qquad \widetilde{\phi}_n(\mathbf{k}^*, s) = e^{-\left(-i\mathbf{k}^* \cdot \mathbf{v}^* + \Lambda^+\right)s} \widetilde{\phi}_n. \tag{28}$$

The second equality of (27) defines the adjoint operator Λ^+.

Other response functions can be defined for more general initial conditions and/or for other variables. These response functions defined here will be called "hydrodynamic" since the variables being considered are hydrodynamic fields, and the initial perturbation is such that the response is entirely hydrodynamic in the long wavelength limit

$$G_{nm}(\mathbf{k}^* = \mathbf{0}, s) = e^{-\lambda_n(0)s} \delta_{nm}. \tag{29}$$

The utility of this special feature is demonstrated in the next section. In general, for $k^* \neq 0$ these "hydrodynamic" response functions will have both hydrodynamic and non-hydrodynamic excitations

$$G_{nm}(\mathbf{k}^*, s) = A_{nm}(k^*) e^{-\lambda_n(k^*)s} + B_{nm}(k^*, s) \tag{30}$$

where $B_{nm}(k^*, s)$ denotes the non-hydrodynamic contribution. It is expected that this part of the spectrum corresponds to faster relaxation than that of the hydrodynamic modes, so that a hydrodynamic description dominates on a long time scale.

4.1. Dispersion relations

The response functions $G_{nm}(\mathbf{k}^*, s)$ describe all possible excitations in response to the initial hydrodynamic perturbation. Generally, it has the form of Eq. (30). These excitations are defined by the singularities of the Laplace transform of $G_{nm}(\mathbf{k}^*, s)$ in the complex plane, e.g.,

$$G_{nm}(\mathbf{k}^*, s) = \frac{1}{2\pi i} \int_c dz e^{zt} \left(zI + \lambda\left(0\right) + M(\mathbf{k}^*, z)\right)_{nm}^{-1}. \tag{31}$$

Here I is the unit matrix, $\lambda\left(0\right)$ is a diagonal matrix whose elements are $\lambda_m(0)$, and $M(\mathbf{k}^*, z)$ is a matrix representing the effects of finite wavevector. The integral is defined as a contour enclosing all singularities of the integrand. These singularities define the spectrum of the response functions, i.e. the solutions $z(k^*)$ to the equations

$$\det\left(z(k^*)I + \lambda\left(0\right) + M(\mathbf{k}^*, z(k^*))\right) = 0. \tag{32}$$

These are the dispersion relations that characterize the physical nature of granular gases on all length and time scales. The specific case of hydrodynamics corresponds to those solutions which continuously approach the above values $\lambda\left(0\right)$ as $k^* \to 0$. The hydrodynamic excitations therefore can be identified by determining the form of the matrix $M(\mathbf{k}^*, z)$ as a function of $\mathbf{k}^*$ and z. This is addressed in the next subsection.

4.2. Green-Kubo Relations

A formal identification of $M(\mathbf{k}^*, z)$ is possible using an analysis paralleling that for normal fluids [15]. In that case the hydrodynamic response functions are constructed from the local conserved quantities (summational invariants), and the conservation laws are used to express M in terms of response functions for the corresponding fluxes. The special choice of initial perturbations considered here in terms of ϕ_n leads to similar "conservation laws"

$$(\partial_s + \lambda_m(0))\phi_m(\mathbf{k}^*, s) + i\mathbf{k}^* \cdot \gamma_m(\mathbf{k}^*, s) = 0, \tag{33}$$

with the "fluxes" given by

$$\gamma_m(\mathbf{k}^*, s) = e^{-(i\mathbf{k}^* \cdot \mathbf{v}^* + \Lambda)s}\gamma_m, \qquad \gamma_m = \mathbf{v}\phi_m \tag{34}$$

Similarly, the adjoint "conservation laws" are found to be

$$(\partial_s + \lambda_m(0))\widetilde{\phi}_m(\mathbf{k}^*, s) - i\mathbf{k}\cdot\widetilde{\gamma}_m(\mathbf{k}^*, s) = \delta_{m2}\widetilde{v}(\mathbf{k}^*, s), \tag{35}$$

with the conjugate fluxes

$$\widetilde{\gamma}_m(\mathbf{k}^*, s) = e^{-\left(-i\mathbf{k}^* \cdot \mathbf{v}^* + \Lambda^+\right)s}\widetilde{\gamma}_m, \qquad \widetilde{\gamma}_m = \mathbf{v}\widetilde{\phi}_m. \tag{36}$$

Also, there is an energy loss source term in (35)

$$\widetilde{v}(\mathbf{k}, s) = e^{-\left(-i\mathbf{k}^* \cdot \mathbf{v}^* + \Lambda^+\right)s}\widetilde{v}, \qquad \widetilde{v} = (1 - P)\left(1 - \alpha^2\right)\frac{\pi}{6}\int d\mathbf{v}_2^* f_{hcs}^*(\mathbf{v}_2^*, t)g^{*3} \tag{37}$$

where P is the projection operator

$$PX = (X, \phi_m)\widetilde{\phi}_m. \tag{38}$$

These conservation laws lead to corresponding equations for the response functions, and expose some important leading order dependence of $M(\mathbf{k}^*, z)$ on k^*. It is clear from (29) that $M(\mathbf{k}^*, z)$ vanishes at $k^* = 0$. Assuming analyticity, the exact forms for the order k^* and $k^{*2} = 0$ can be determined. These give the Euler and Navier-Stokes forms for the hydrodynamics, respectively. The method is outlined in the Appendix with the result

$$M(k, z) = ikM^{(1)} + k^2 M^{(2)}(z) + .. \tag{39}$$

$$M_{mn}^{(1)} = \left(\widetilde{\phi}_m, \gamma_n^i\right) \tag{40}$$

$$M_{mn}^{(2)}(z) = \widehat{k}_i\widehat{k}_j \int_0^\infty ds e^{-zs}\left[\left(\widetilde{\gamma}_q^j, e^{-\Lambda s}(1 - P)\gamma_m^i\right) + \delta_{m2}\int_0^s ds\left(\widetilde{v}, e^{-\Lambda\left(s-s'\right)}v_j e^{-\Lambda s'}\gamma_n^i\right)\right] \tag{41}$$

The terms $M^{(1)}$ at linear order in k^* are just the coefficients of the first order gradients in the linearized Euler equations. The terms of second order contain information about all finite wavevector excitations through its dependence on z. As discussed above, the special hydrodynamic excitations are those arising from the eigenvalues of $\lambda(0)$ at $k^* = 0$, i.e. the solutions to

$$\det\left(z_h(k)I + \lambda(0) + ikM^{(1)} + k^2 M^{(2)}(z_h(0))\right) \tag{42}$$

The contributions to these solutions from $M^{(2)}(-z_h(0))$ define the transport coefficients at Navier-Stokes order. From Eq. (41) there are terms that are time integrals of the flux - flux response function. This is the generalization of Green-Kubo expressions for a granular gas [13, 6]. In addition, the second term of (41) gives transport coefficients associated with the cooling rate that have no analogue for normal gases. The results given here agree with those obtained by other methods, such as the Chapman-Enskog solution to the Boltzmann equation and a perturbation theory calculation of the eigenvalues in Eq. (17)

5. CONCLUSIONS

The primary results here have been the identification of hydrodynamic excitations in the spectrum of the linearized Boltzmann operator, and using the corresponding eigenfunctions to construct hydrodynamic response functions. While hydrodynamics can be obtained for granular gases in other ways (e.g., the Chapman-Enskog expansion), the approach via the spectrum is more direct and free of many conceptual stumbling blocks for the Chapman-Enskog method. The response functions are special in the sense that they involve the functions ϕ_n which obey "conservation laws" relating them to the gradients of fluxes. They therefore have a close analogy to the response functions for normal gases constructed from local conserved densities. As a consequence, a purely algebraic manipulation of the response functions leads to an identification and representation of transport coefficients in terms of Green-Kubo expressions.

The Green-Kubo representation may be a useful alternative form for direct numerical evaluation of the transport coefficients [16]. More generally, the analysis given here based on the Boltzmann equation points the direction for a similar but formally exact derivation of Green-Kubo relations from the Liouville equation. That analysis is in progress.

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A. APPENDIX: SPECTRUM OF RESPONSE FUNCTIONS

Using the conservation laws response functions are found to obey the equations

$$(\partial_s + \lambda_m(0))G_{nm}(\mathbf{k}^*, s) + ik_i G_{nm}^i(\mathbf{k}^*, s) = 0 \tag{43}$$

$$(\partial_s + \lambda_n(0))G_{nm}(\mathbf{k}^*, s) + ik_i H_{nm}^i(\mathbf{k}^*, s) = \delta_{n2}\left(\widetilde{\nu}, \phi_m(\mathbf{k}^*, s)\right) \tag{44}$$

$$(\partial_s + \lambda_n(0))G_{nm}^i(\mathbf{k}^*, s) + ik_j G_{nm}^{ji}(\mathbf{k}^*, s) = \delta_{n2}\left(\widetilde{\nu}, \gamma_m^i(\mathbf{k}^*, s)\right) \tag{45}$$

where $G_{nm}^i(\mathbf{k}^*, s)$, $H_{nm}^i(\mathbf{k}^*, s)$, and $G_{nm}^{ji}(\mathbf{k}^*, s)$ are response functions involving the fluxes

$$G_{nm}^i(\mathbf{k}^*, s) = \left(\widetilde{\phi}_n, \gamma_m^i(\mathbf{k}^*, s) \right) = \left(\widetilde{\phi}_n(\mathbf{k}^*, s), \gamma_m^i \right), \qquad H_{nm}^i(\mathbf{k}^*, s) = \left(\widetilde{\gamma}_n^i, \phi_m(\mathbf{k}^*, s) \right), \qquad G_{nm}^{ji}(\mathbf{k}^*, s) = \left(\widetilde{\gamma}_n^j, \gamma_m^i(\mathbf{k}^*, s) \right), \tag{46}$$

and the source terms $S_{nm}(\mathbf{k}^*, s)$ and $S_{nm}^i(\mathbf{k}^*, s)$ are

$$S_{nm}(\mathbf{k}^*, s) = \delta_{n2} \left(\widetilde{\nu}, \phi_m(\mathbf{k}^*, s) \right), \qquad S_{nm}^i(\mathbf{k}^*, s) = \delta_{n2} \left(\widetilde{\nu}, \gamma_m^i(\mathbf{k}^*, s) \right). \tag{47}$$

Define the Laplace transform of a function $F(s)$ by

$$\widehat{F}(z) = \int_0^\infty ds\, e^{-zs} F(s). \tag{48}$$

Then the Laplace transformed equations (43) - (44) are (in an obvious matrix notation where I is the identity matrix and $\lambda(0)$ is a diagonal matrix with elements $\lambda_n(0)$)

$$\widehat{G}(\mathbf{k}^*, z)(z + \lambda(0)) + ik_i^* \widehat{G}^i(\mathbf{k}^*, z) = I \tag{49}$$

$$(z + \lambda(0))\widehat{G}(\mathbf{k}^*, z) + ik_i^* \widehat{H}^i(\mathbf{k}^*, z) = I + S(\mathbf{k}^*, z) \tag{50}$$

$$(z + \lambda(0))\widehat{G}^i(\mathbf{k}^*, z) + ik_j^* \widehat{G}^{ji}(\mathbf{k}^*, z) = \left(\widetilde{\phi}_n, \gamma_m^i \right) + S^i(\mathbf{k}^*, z) \tag{51}$$

In this notation the matrix $M(\mathbf{k}^*, z)$ defining the dispersion relations is defined by

$$\widehat{G}(\mathbf{k}^*, z) = \left[(zI + \lambda(0)) + M(\mathbf{k}^*, z) \right]^{-1}.$$

Equation (49) then gives

$$M(\mathbf{k}^*, z) = ik_i \widehat{G}^{-1}(\mathbf{k}^*, z)\widehat{G}^i(\mathbf{k}^*, z). \tag{52}$$

Multiplying by $(z + \lambda(0))$ and its inverse gives

$$\begin{aligned}
M(\mathbf{k}^*, z) &= ik_i^* \left((z + \lambda(0))\widehat{G}(\mathbf{k}^*, z) \right)^{-1} ((z + \lambda(0)))\, \widehat{G}^i(\mathbf{k}^*, z) \\
&= ik_i^* \left(I - ik_i^* \widehat{H}^i(\mathbf{k}^*, z) + S(\mathbf{k}^*, z) \right)^{-1} \left(-ik_j^* \widehat{G}^{ji}(\mathbf{k}^*, z) + \left(\widetilde{\phi}, \gamma^i \right) + S^i(\mathbf{k}^*, z) \right)
\end{aligned} \tag{53}$$

where Eqs. (50) and (51) have been used in the second equality. This is the desired form expressing $M(\mathbf{k}^*, z)$ in terms of flux response functions and with the dominant k^* dependence explicit.

Further simplifications occur for small k^*. If only terms through k^{*2} are to be retained (Navier-Stokes order) then only $\widehat{H}^i(\mathbf{0}, z)$ is require in (53). Furthermore $S(\mathbf{k}^*, z)$ is of order k^{*2} from symmetry and does not contribute to this order, and $S^i(\mathbf{0}, z) = 0$. Therefore

$$M(\mathbf{k}^*, z) \to ik_i \left(\widetilde{\phi}_p, \gamma_m^i \right) + k_i k_j \left(\widehat{G}^{ji}(\mathbf{0}, z) - \widehat{H}^j(\mathbf{0}, z) \left(\widetilde{\phi}, \gamma^i \right) + \frac{\partial}{\partial ik_j} \left(S^i(\mathbf{k}, z) \right)_{k=0} \right). \tag{54}$$

Writing explicitly the transforms on the right side as time integrals, and using the identity

$$G_{qm}^{ji}(\mathbf{0}, s) - H_{qp}^j(\mathbf{0}, s) \left(\widetilde{\phi}_p, \gamma_m^i \right) = \left(\widetilde{\gamma}_q^j(\mathbf{k}, s), (1 - P)\, \gamma_m^i \right), \tag{55}$$

gives Eqs. (39) - (41) of the text.